Detection of hidden regimes in stochastic cyclostationary time series

Volkmar Wirth*
Meteorological Institute, University of Munich, Munich, Germany
(Received 9 October 2000; published 28 June 2001)

Stationary stochastic time series with nonlinear dynamics can feature a probability density function (PDF) with distinct local maxima associated with distinct regimes. For nonstationary time series, on the other hand, such regimes are not necessarily reflected in the shape of the PDF. This occurs when the duration of a regime is too short for the PDF to adjust, and such a regime is called a ''hidden'' regime. This paper presents an algorithm that allows one to detect hidden regimes in cyclostationary stochastic Markovian time series. The method involves analysis of an appropriately windowed time series, from which the drift and diffusion coefficients of the associated Fokker-Planck equation are estimated. The success of the algorithm is illustrated using synthetic time series with both additive and multiplicative noise.

DOI: 10.1103/PhysRevE.64.016136 PACS number(s): 02.50.−r

I. INTRODUCTION

Stochastic time series are generally characterized by a deterministic part on the one hand and a random part on the other hand. Nonlinearity of the deterministic part may give rise to phenomena such as multiple regimes and noise induced transitions [1]. For stationary time series multiple regimes are associated with multiple local maxima of the probability density function (PDF). For nonstationary time series the situation is more complex: if a regime is transient and of relatively short duration, the PDF may not have time to adjust in a quasistationary fashion. Such a regime will be called a ''hidden'' regime, because it is not associated with a relative maximum of the PDF as in the stationary case.

Cyclostationary processes can be viewed as intermediate between stationary and nonstationary. They are of particular interest to many natural sciences, since in nature many processes are cyclostationary rather than stationary. For instance, the diurnal cycle or the seasonal cycle often represent a dominant external forcing in biological or geophysical systems. Moreover, the period T is often known *a priori*, because the process is governed by external parameters whose variation can be measured.

With such applications in mind, we study in this paper Markovian time series, which are known to be cyclostationary *a priori*. More specifically we will show that it may be possible to detect hidden regimes from such time series using a straightforward technique. The algorithm will be presented and illustrated by means of examples involving synthetic stochastic time series. For simplicity we restrict ourselves to time series with one variable, but the extension of the basic idea to N variables is straightforward. Otherwise the presentation is kept rather general, as the method may find potential applications in a wide range of areas. Remarks concerning the application to real problems including a specific example can be found at the end of the paper.

*Present address: Institute for Physics of the Atmosphere, University of Mainz, Mainz, Germany.

II. DEFINITIONS AND BASIC RELATIONS

Consider the dynamics of a continuous one-dimensional Markovian system governed by the following Langevin equation for its variable $x(t)$,

$$\frac{dx}{dt}=G(x,t)+g(x,t)Z(t), \tag{1}$$

where t denotes time and $Z(t)$ is Gaussian white noise satisfying

$$\langle Z(t)\rangle=0, \tag{2}$$

$$\langle Z(t)Z(t')\rangle=Q\,\delta(t-t'). \tag{3}$$

The angles denote an ensemble mean. The functions $G(x,t)$ and $g(x,t)$ represent the dynamics of the system and are considered to be externally specified. Additive noise corresponds to the special case $\partial g/\partial x\equiv 0$, otherwise the noise is multiplicative. The associated Fokker-Planck equation, which describes the evolution of the probability density function $p(x,t)$, reads

$$\frac{\partial p}{\partial t}=-\frac{\partial}{\partial x}(Ap)+\frac{1}{2}\frac{\partial^2}{\partial x^2}(Bp). \tag{4}$$

Here, $A(x,t)$ and $B(x,t)$ denote the drift and diffusion coefficients, respectively, which are defined as

$$A(x,t)=\lim_{\tau\to 0}\frac{1}{\tau}\langle\tilde{x}(t+\tau)-\tilde{x}(t)\rangle, \tag{5}$$

$$B(x,t)=\lim_{\tau\to 0}\frac{1}{\tau}\langle[\tilde{x}(t+\tau)-\tilde{x}(t)]^2\rangle, \tag{6}$$

where $\tilde{x}(t)$ is a solution of Eq. (1) with $\tilde{x}(t)=x$ (e.g., see Ref. [2]). When Eq. (1) is interpreted in the Ito sense, the following relations between the coefficients of the Langevin equation (1) and those of the Fokker-Planck equation (4) hold:

$$A(x,t)=G(x,t), \tag{7}$$

$$B(x,t)=Qg^2(x,t). \tag{8}$$

The process is called stationary if neither of the coefficients A, B, G, and g explicitly depend on time. In that case (4) has a stationary solution satisfying

$$\frac{\partial J}{\partial x}=\text{const} \tag{9}$$

with

$$J(x)=Ap(x)-\frac{1}{2}\frac{\partial}{\partial x}[Bp(x)]. \tag{10}$$

Assuming that the PDF is well behaved and goes to zero as $|x|\to\infty$, this solution can be written as

$$p(x)=p_s(x)\equiv\mathcal{N}e^{-U(x)}, \tag{11}$$

where $\mathcal{N}$ is a normalization constant and

$$U(x)=-\int_{x_0}^{x}\frac{2}{B(x')}\left[A(x')-\frac{1}{2}\frac{\partial B(x')}{\partial x}\right]dx' \tag{12}$$

is the so-called stochastic potential (cf. Ref. [1], The value of x_0 is arbitrary; once fixed it determines the value of $\mathcal{N}$ through the normalization condition. A local minimum of $U(x)$ corresponds to a local maximum of $p_s(x)$. We define each local maximum of $p_s(x)$ to be associated with a ''regime,'' and multiple local maxima correspond to multiple regimes. Note that the definition of the ''stationary PDF'' $p_s(x)$ and the definition of regimes involves, via Eqs. (7), (8), (11), and (12), only the functions $G(x,t)$ and $g(x,t)$ representing the underlying dynamics of the system. These definitions can be generalized to nonstationary time series.

A process is called cyclostationary, when both $G(x,t)$ and $g(x,t)$ are periodic in time with period T, i.e., when

$$G(x,t+T)=G(x,t), \tag{13}$$

$$g(x,t+T)=g(x,t), \tag{14}$$

for all times t. Introducing the phase

$$\phi(t)=\text{mod}(t,T), \tag{15}$$

the coefficients G, g, A, and B can be expressed as functions of x and ϕ. Consistent with the above we define

$$U(x,\phi)=-\int_{x_0}^{x}\frac{2}{B(x',\phi)}\left[A(x',\phi)-\frac{1}{2}\frac{\partial B(x',\phi)}{\partial x}\right]dx', \tag{16}$$

and

$$p_s(x,\phi)\equiv\mathcal{N}e^{-U(x,\phi)}. \tag{17}$$

Again, we speak about multiple regimes when $p_s(x,\phi)$ has multiple local maxima with regard to its variable x. A regime is called transient when it exists only during a fraction of the period T. Generally, the PDF $p(x,\phi)$ of a cyclostationary process at phase ϕ is not given by $p_s(x,\phi)$,

$$p(x,\phi)\neq p_s(x,\phi), \tag{18}$$

because Eq. (11) is valid only for stationary processes. A ''hidden regime'' is defined to be a transient local maximum (with respect to x) of $p_s(x,\phi)$ which is *not* associated with a local maximum of the corresponding PDF $p(x,\phi)$.

Broadly speaking a hidden regime can be expected to occur when the internal evolution of the system is too slow to follow the external change. More formally, for smoothly varying G and g we introduce the external time scale

$$T_{\text{ext}}=\min\left(\frac{G}{\partial G/\partial t},\quad \frac{g}{\partial g/\partial t}\right) \tag{19}$$

as the smaller of the two time scales on which G and g vary, furthermore the dynamical time scale

$$T_{\text{dyn}}=\min\left(\frac{x}{G},\frac{x^2}{Qg^2}\right) \tag{20}$$

as the time scale associated with the internal dynamics of the system. A nontrivial cyclostationary process satisfies $T_{\text{ext}}\leqslant T$. The occurrence of hidden regimes can be expected for $T_{\text{ext}}\ll T_{\text{dyn}}$. On the other hand, for $T_{\text{ext}}\gg T_{\text{dyn}}$ the external change is much slower than the internal dynamics such that the system adjusts in a quasistationary fashion and $p(x,\phi)\approx p_s(x,\phi)$.

It is the goal of the present paper to describe a method that allows to uncover hidden regimes in cyclostationary time series.

III. METHOD

Our method is based on an algorithm published recently by Siegert *et al.* [3]. The authors showed that it is possible to compute the drift and diffusion coefficients $A(x)$ and $B(x)$ from a stationary time series $x(t)$ through recourse to the Markovian property and the basic definitions (5) and (6). In this approach stationarity allows to replace the ensemble mean by an average over the time series. Here, we generalize the method to cyclostationary time series. A time window

$$w(\phi)=\begin{cases}1, & 0\leqslant\phi_1<\phi<\phi_2\leqslant 1\\ 0, & \text{else,}\end{cases} \tag{21}$$

is defined, where ϕ_1 and ϕ_2 are chosen such that the external coefficients $G(x,\phi)$ and $g(x,\phi)$ vary only marginally during the time interval $\phi_1<\phi<\phi_2$, i.e., while the window is open $(w=1)$. As a typical application we envisage w to be equal to one while the hidden regime exists, and zero otherwise. The windowed time series $x^w(t)$ is defined to be a compressed version of $x(t)$ consisting only of those elements for which $w=1$. Although $x^w(t)$ is not necessarily stationary, it is possible to apply the algorithm of Siegert *et al.* [3] to obtain the coefficients $A^w(x)$ and $B^w(x)$. Here, the dependence on phase ϕ has been dropped, since it is marginal owing to the definition of the window. The corresponding windowed potential $U^w(x)$ using Eq. (12) with A and B re-

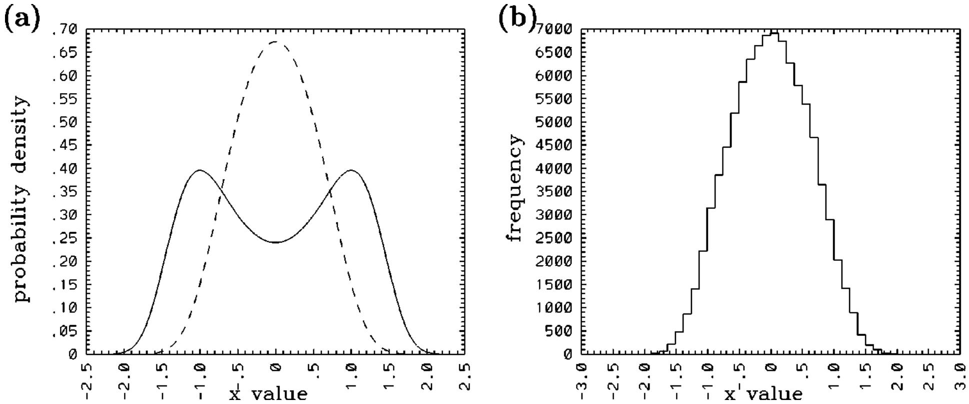

FIG. 1. Numerical example with additive noise: (a) Stationary PDF $p_s(x)$ for a time series with $G=G_\nu$ from Eq. (28) with $\nu=1$ (solid line) and $\nu=0$ (dashed line); (b) histogram of the windowed time series $x^w(t)$ for a cyclostationary process that switches between $\nu=1$ within the window and $\nu=0$ outside the window.

placed by A^w and B^w yields information about hidden regimes (if present). The stationary PDF of the windowed time series, given by

$$p_s^w(x)=\mathcal{N}e^{-U^w(x)}, \tag{22}$$

is the hypothetical PDF that one would observe if the time series were stationary and characterized by the coefficients A^w and B^w. Note that the time series does not have to be stationary for Siegert *et al.*'s algorithm to be valid. The relations (5), (6), (7), and (8) are all valid for general nonstationary time series. The only important requirement is to collect enough statistical information such that the ensemble mean in Eqs. (5) and (6) can be replaced through an average over suitable sections of the time series. An average over the windowed time series achieves this goal, since by definition the window filters out the same section from each cycle, allowing to accumulate the relevant statistical information over many cycles.

We apply the algorithm to a discrete finite time series, i.e., to

$$x_i=x(t_i), \quad i=1,2,\ldots n, \tag{23}$$

which is meant to approximate the continuous infinite time series discussed so far. The time step $\Delta t=t_{i+1}-t_i$ is assumed to satisfy $\Delta t\ll\min(T,T_{\mathrm{ext}},T_{\mathrm{dyn}})$ such that the basic features of the continuous time series are well resolved. Furthermore, we define the time series of increments

$$dx_i=x_{i+1}-x_i. \tag{24}$$

The discrete time series can be represented as a sequence of points

$$i\rightarrow(x_i,dx_i) \tag{25}$$

in the two-dimensional x-dx plane. Appropriately binning the data into x bins, the windowed coefficients $A^w(x)$ and $B^w(x)$ are estimated as follows:

$$A^w(x)\approx\frac{1}{K}\sum_{k=1}^{K} dx_k^w(x), \tag{26}$$

$$B^w(x)\approx\frac{1}{K}\sum_{k=1}^{K}[dx_k^w-A^w(x)]^2, \tag{27}$$

where dx^w represents the windowed time series of the increments, and the sum extends over the K points within the bin corresponding to the value x.

IV. EXAMPLES

We illustrate our method with the help of synthetically generated time series. In particular we will show how it allows to detect hidden regimes.

A. Additive noise

Consider a family of stationary processes described by Eq. (1) with $g=1$, $Q=0.01$ and

$$G(x)=G_\nu(x)\equiv0.01[(2\nu-1)x-x^3]. \tag{28}$$

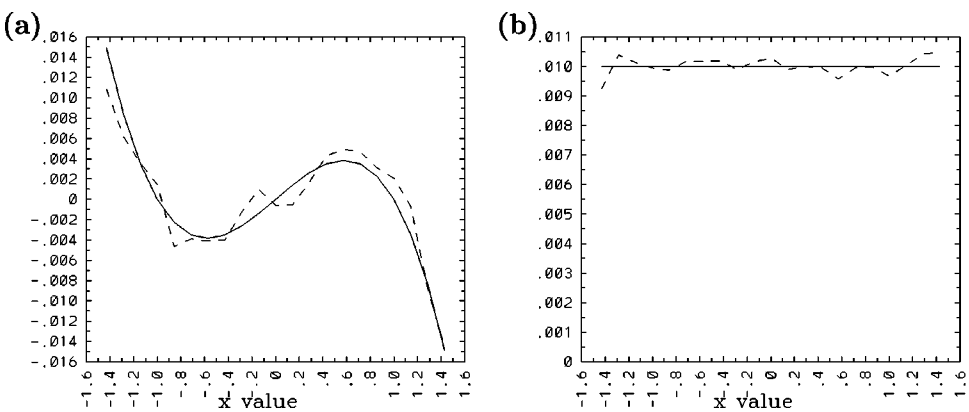

FIG. 2. Parameters of the windowed time series in the example with additive noise: (a) drift coefficient $A^w(x)$, and (b) diffusion coefficient $B^w(x)$. In both panels, the solid line delineates the hypothetical curve, while the dashed line represents the reconstruction.

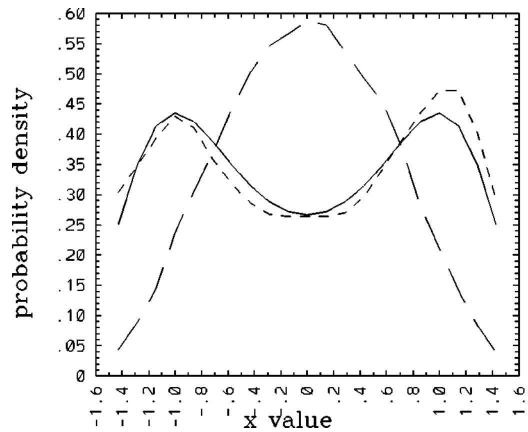

FIG. 3. Cyclostationary time series with additive noise: true stationary PDF $p_s^w(x)$ (solid), reconstructed stationary PDF (short dashes), and empirical PDF $p^w(x)$ (long dashes) for the windowed time series.

The corresponding potential (12) becomes

$$U(x) = -\frac{2}{Q}\int_{x_0}^{x} G_\nu(x')\,dx' = \frac{1}{Q}\left[\frac{1}{2}x^4 - (2\nu-1)x^2\right] + \text{const.} \tag{29}$$

For $\nu=+1$ this function has two local minima at $x=\pm 1$, while for $\nu=0$ there is only one minimum at $x=0$. Correspondingly, the stationary PDF $p_s(x)$ computed from Eq. (11) has two local maxima for $\nu=+1$ [solid line in Fig. 1(a)], while there is only one local maximum for $\nu=0$ (dashed line). In other words, there are two regimes for $\nu=+1$, but only one regime for $\nu=0$.

In order to generate a cyclostationary time series with hidden regimes, we choose $T=200$ and switch between $\nu=+1$ and $\nu=0$ using $\nu=w$ and $w(\phi)$ as in Eq. (21) with $\phi_1=0.45T$ and $\phi_2=0.55T$. This means that there is one regime for 90% of the time, but there are two regimes within the window $\phi_1<\phi<\phi_2$. The present choice of parameters yields $x\sim 1$ and, hence, $T_{\rm dyn}\sim 100$. The time scale of the externally imposed transience owing to the discontinuous changes of G is estimated as $T_{\rm ext}\sim\phi_2-\phi_1=0.1T=20$, i.e., $T_{\rm ext}\ll T_{\rm dyn}$. We expect that the two regimes within the window are hidden.

The discretized version of Eq. (1) is simulated numerically by performing 5000 cycles with a time step of $\Delta t=1$, i.e., by performing 200×5000 time steps. As anticipated, the windowed time series $x^w(t)$ has a unimodal distribution $p^w(x)$ [Fig. 1(b)]: it does not reveal a hint of bimodality, although the true stationary PDF $p_s^w(x)$ [solid line in Fig. 1(a)] is clearly bimodal. This discrepancy corresponds to the unequality (18)

We now arrange the discrete values of the windowed time series into bins of width $\Delta x=0.14$ and estimate $A^w(x)$ and $B^w(x)$ using Eqs. (26) and (27). Only those bins are considered for which $K\geqslant 5$, which restricts the range of values of x for which the coefficients can be reconstructed to $|x|\leqslant 1.6$ in the present example. As demonstrated in Fig. 2, the reconstructed functions (dashed lines) are in close agreement with the hypothetical curves obtained from Eqs. (7) and (8) and the a priori knowledge of G and g (solid lines).

Finally we compute $p_s^w(x)$ from Eq. (22) using the reconstructed coefficients $A^w(x)$ and $B^w(x)$ in the expression for $U^w(x)$. The reconstructed PDF is displayed as the short dashes in Fig. 3. Comparison with the true stationary PDF $p_s^w(x)$ (solid line) based on the *a priori* knowledge of G and g demonstrates that the method works successfully: the bimodal structure of the reconstructed function is in sharp contrast to the unimodal empirical PDF $p^w(x)$ of the windowed time series (long dashes in the same figure). In other words, the hidden regimes, which are not noticeable in the empirical PDF, have been recovered with the algorithm.

B. Multiplicative noise

As a second example we consider a cyclostationary time series with the same parameters as above except that

$$G(x) = -0.001x \tag{30}$$

and

$$g(x) = g_\nu(x) \equiv 1 + \frac{\nu}{2}\left(e^{-x^2}-1\right) \tag{31}$$

with $\nu=w(\phi)$. This corresponds to multiplicative noise, and the bimodality within the window (i.e., for $\nu=1$) arises from the inhomogeneity of the noise term. Figure 4(a) depicts the stationary PDF $p_s(x)$ inside ($\nu=1$, solid line) and outside

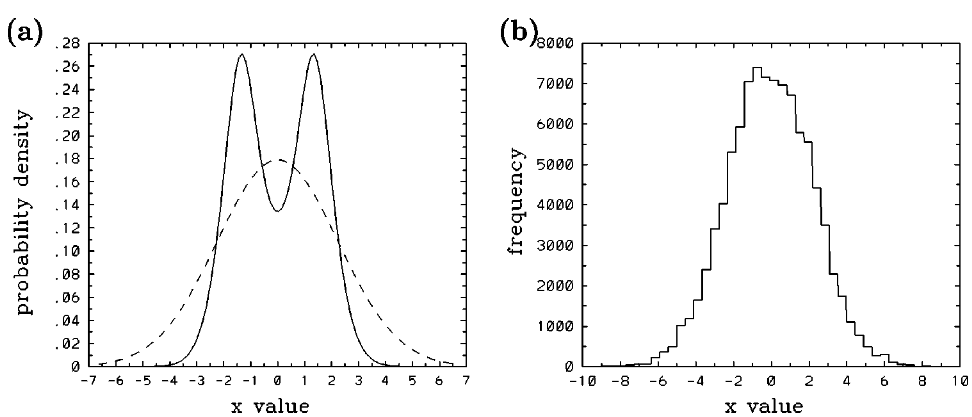

FIG. 4. Numerical example with multiplicative noise: (a) Stationary PDF $p_s(x)$ for a time series with external forcing according to Eq. (31) with $\nu=1$ (solid line) and $\nu=0$ (dashed line); (b) histogram of the windowed time series $x^w(t)$ for a cyclostationary process which switches between $\nu=1$ within the window and $\nu=0$ outside the window.

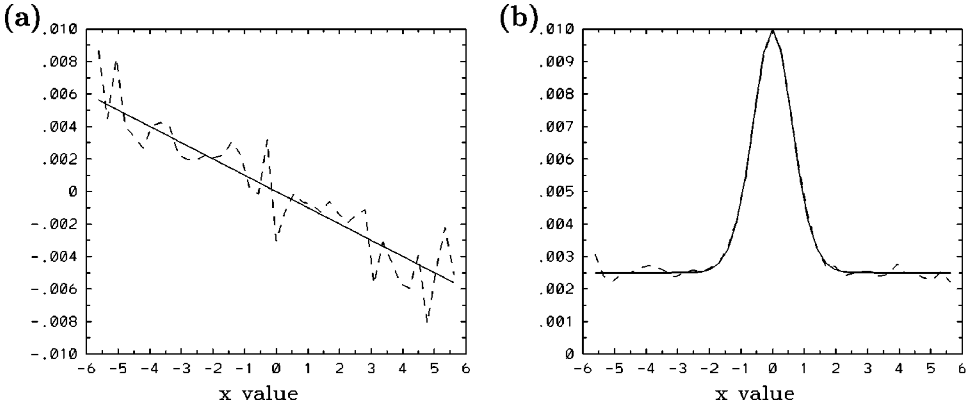

FIG. 5. Parameters of the windowed time series in the example with multiplicative noise: (a) drift coefficient $A^w(x)$, and (b) diffusion coefficient $B^w(x)$. In both panels, the solid line delineates the hypothetical curve, while the dashed line represents the reconstruction.

($\nu = 0$, dashed line) the window. The histogram of x values within the window [Fig. 4(b)] is clearly unimodal; thus, the two regimes corresponding to the two maxima of the solid curve in Fig. 4(a) are ''hidden.''

Again, the algorithm (with $\Delta x = 0.28$) successfully reconstructs the parameters $A^w(x)$ and $B^w(x)$ (Fig. 5). Note in particular that the inhomogeneity of the diffusion coefficient $B^w(x)$ in Fig. 5(b) [corresponding to the noise term $g(x)$] is reconstructed well. As a consequence, the reconstructed stationary PDF (short dashes in Fig. 6) reproduces well the bimodal behavior of the true stationary $p_s^w(x)$ (solid line), in sharp contrast to the empirical PDF $p^w(x)$ of the windowed time series (long dashes).

C. Statistical error

Obviously the reconstruction method is subject to statistical error that increases as the length of the time series decreases. In order to estimate the uncertainty regarding the reconstructed PDF we consider an ensemble of ten time series corresponding to ten different realizations of $Z(t)$ in Eq. (1). The curves in Fig. 7 represent the reconstructed stationary PDF's $p_s^w(x)$ for the example from Sec. IV A.When each time series consists of 5000 cycles [Fig. 7(a)], all curves clearly show a bimodal behavior indicating the existence of two (hidden) regimes (cf. Fig. 3). On the other hand, when the length of each time series is reduced to 500 cycles [Fig. 7(b)], the spread of the curves is significantly larger, and some of the lines do not represent the proper qualitative behavior. Yet, even here all curves deviate strongly from the empirical PDF $p^w(x)$ with its single maximum at $x \approx 0$ (long dashes in Fig. 3(b), indicating that the latter does not represent the underlying dynamics.

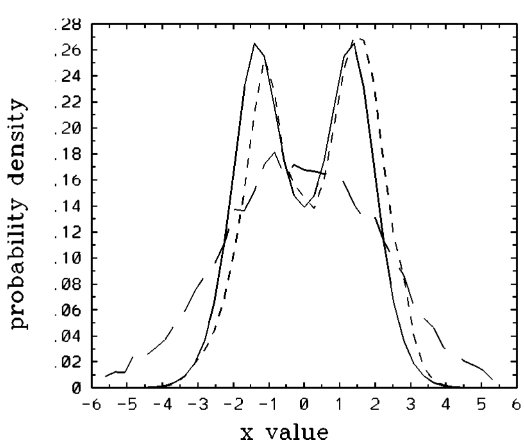

FIG. 6. Cyclostationary time series with multiplicative noise: true stationary PDF $p_s^w(x)$ (solid), reconstructed stationary PDF (short dashes), and empirical PDF $p^w(x)$ (long dashes) for the windowed time series.

V. GENERALIZATION

A closer examination reveals that our method does not make explicit use of the property of cyclostationarity. It can, therefore, be applied to more general time series. The key requirement to be met is the existence of a phenomenon (like, e.g., the existence of multiple hidden regimes) that recurs intermittently, but it does not have to recur with a fixed period T. Furthermore, it is necessary that the occurrence of the phenomenon is known or can be inferred by some means. Examples may be found in biology or geophysics, where certain processes require the presence of sunlight or the temperature to exceed a threshold. As long as the rate of change of G and g is small during the time span of the phenomenon, a windowed time series can be constructed by concatenating the relevant sections of the full time series. The remaining part of the analysis carries over without change.

VI. SUMMARY AND POTENTIAL APPLICATION

A method was presented that allows one to detect hidden regimes in cyclostationary Markovian stochastic time series. The method involves the construction of appropriately windowed time series, from which the drift and diffusion coefficients of the associated Fokker-Planck equation are determined. The method was demonstrated to work well with synthetic time series for both additive and multiplicative noise.

When applying the algorithm to real data one has to keep in mind the assumptions made in the present work. These include the noise to be white and the time series to be long enough. We studied the sensitivity of our results to the length of the time series and found that the statistical error may prevent the analysis if there is not enough data. Whether available data are sufficient to extract the relevant informa-

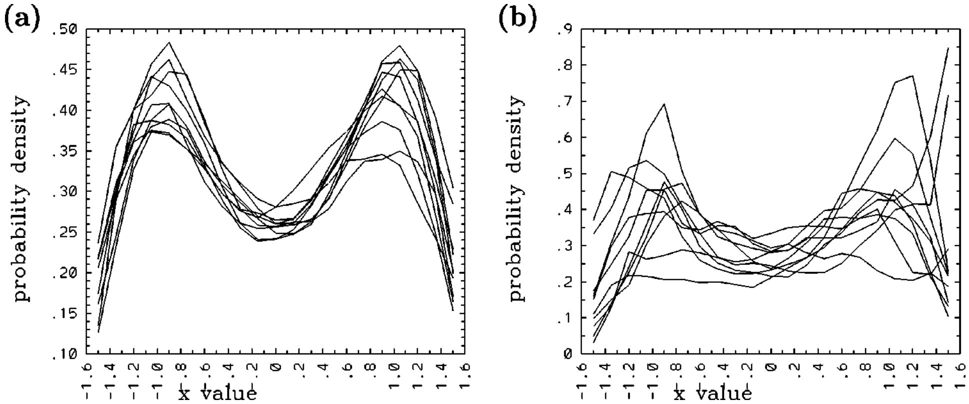

FIG. 7. Reconstructed PDF for ten different realizations of the stochastic time series in the example with additive noise. The time series has 5000×200 time steps in (a), and 500×200 time steps in (b).

tion is a question that has to be answered on a case to case basis. When the noise is correlated, it may be necessary to reduce the time series by appropriately averaging prior to the analysis. Recently, Egger [4] suggested a method showing how climatologic data can be investigated in the framework of the Fokker-Planck equation. A similar method may turn out helpful in the current context. Furthermore, the method requires the definition of a ''window.'' Often this is straightforward, like in the example given below. In other cases, however, it may not be entirely clear which part of the cycle to include into the window. In such cases experimentation with different windows may prove necessary and useful.

As mentioned in the introduction, we envisage a number of interesting applications in biology and geophysics. As an example, which is relevant to regional climate, we note the interaction between soil moisture and precipitation. During most of the year, precipitation is determined by the atmospheric larger-scale circulation, and soil moisture simply responds in a passive manner. However, during the summer season precipitation is controlled by moist convection, which in turn is significantly influenced by soil moisture. This interaction may lead to multiple regimes under certain conditions (Rodriguez-Iturbe *et al.*, [5]; Entekhabi *et al.*, [6]). The system may, therefore, be associated with multiple regimes during the summer season (a dry regime and a moist regime), while during the rest of the year only one regime prevails. If the duration of the summer season is short enough and if the intrinsic dynamical time scales of the governing mechanisms (i.e., those leading to changes in soil moisture) are slow enough, these multiple regimes may be hidden. In other words, the empirical PDF of soil moisture during summer would be unimodal although the underlying processes favor either a dry or a moist regime. The windowed time series in the present example consists of the summer months during which the precipitation is dominated by moist convection. Our method allows one, in principle, to detect these hidden dry and moist regimes providing information about the processes responsible for soil moistening and drying. This information cannot be inferred from simpler statistics such as the empirical PDF.

[1] W. Horsthemke and R. Lefever, *Noise-Induced Transitions. Theory and Applications in Physics, Chemistry, and Biology* (Springer-Verlag, Berlin, 1984), p. 318.
[2] C. W. Gardiner, *Handbook of Stochastic Methods*, 2nd ed. (Springer, Berlin, 1985).
[3] S. Siegert, R. Friedrich, and J. Peinke, Phys. Lett. A **243**, 275 (1998).
[4] J. Egger, Clim. Dyn. (to be published).
[5] I. Rodriguez-Iturbe, D. Entekhabi, and R. L. Bras, Water Resour. Res. **27**, 1899 (1991).
[6] D. Entekhabi, I. Rodriguez-Iturbe, and F. Castelli, J. Hydrol. **184**, 3 (1996).